Elongational behavior of gelled propellant simulants

A. L. Yarin, a) E. Zussman, and A. Theron

Faculty of Mechanical Engineering, Technion-Israel Institute of Technology, Haifa 32000, Israel

S. Rahimi, Z. Sobe, and D. Hasan

RAFAEL, MANOR Propulsion and Explosive Systems Division, P.O. Box 2250(M1), Haifa 31021, Israel

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Synopsis

An elongational rheometer is used to study the rheological behavior of gelled propellant simulants in uniaxial elongational flow. In simple shear such fluids typically exhibit a shear thinning behavior which could be described by a power-law constitutive equation. Knowledge of the elongational behavior of these fluids is important for understanding the processes of their atomization and spray formation. The results of the present work demonstrated that the three-dimensional power-law model permits description of uniaxial elongation of these fluids as well. Moreover, the values of the rheological parameters of the tested fluids measured in simple shear and in uniaxial elongation agree fairly closely. Therefore the elongational behavior of gelled propellant simulants can be inferred from their shear behavior. © 2004 The Society of Rheology. [DOI: 10.1122/1.1631423]

I. INTRODUCTION

Power fuel gels and gelled oxidizers are considered potential candidates for rocket propulsion. Both organic and inorganic compounds are used for propellant gellation, and only inorganics for oxidizers. Gelled propellants are based, for example, on hydrazine (N₂H₄) and kerosene, gelled oxidizers, for example, on hydrogen peroxide (H₂O₂). Additives, e.g. aluminum or boron particles, could be suspended in gelled propellants in order to enhance their power value and volumetric specific impulse. As an example of the gellants used, silica and Carbopol could be mentioned.

Since gelled propellants and oxidizers are explosive, toxic and corrosive materials, a need arises for simulants, which would permit safer testing of their flow and atomization behavior. In view of the need for mutual matching of the gelled fuel, oxidizer and simulant, rheological characterization of the three groups is called for.

Rheological measurements for gelled fuels, oxidizers, and their simulants were conducted in Gupta et al. (1986), Rapp and Zurawski (1988), Varghese et al. (1995), and Rahimi et al. (2001). In all these studies simple shear flow was employed. The results revealed pronounced shear-thinning, pseudoplastic behavior of the shear viscosity.

The generalized Herschel-Bulkley constitutive model [Macosko (1994)] was found to fit the simple shear measurements of the gels.

a)Author to whom correspondence should be addressed; electronic mail: meralya@yarin.technion.ac.il

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Since knowledge of the elongational behavior of gelled propellants, oxidizers and their simulants is of importance in understanding their atomization, spray formation and combustion, a natural question arises: To what extent is rheological behavior in simple shear representative of that in elongation? Several examples where the three-dimensional power law is valid for both simple shear and uniaxial elongation are available. Entov et al. (1980) and Yarin (1993) demonstrated that this is the case for a number of suspensions of γ-Fe₂O₃ particles in oil, and for aqueous suspensions of clay. The aim of the present work is to check whether it would also be true for gelled propellant simulants. For this purpose an elongational viscometer was used to characterize the elongational flow behavior, while shear viscosity was measured with a rotational viscometer.

In the present work, the elongational viscometer was based on the capillary thinning of a liquid thread. The idea of the device dates back to the early 1980s [Schummer and Tebel (1980), (1982), (1983); Bazilevsky et al. (1981)] when it was proposed for measurement of the relaxation time of dilute and semidilute polymer solutions. It was systematically applied for measurement of the elongational behavior of solutions of flexible polymers in Bazilevsky et al. (1990a, b, 1994, 1997, 2001), Liang and Mackley (1994), Kolte and Szabo (1999), Stelter et al. (1999, 2000, 2002), Wunderlich et al. (2000), and Anna and McKinley (2001). A recent review of such applications can be found in McKinley (2000). McKinley and Tripathy (2000) showed that this device is capable of measuring the Trouton viscosity of Newtonian fluids. Stelter et al. (2002) applied it to solutions of rigid polymers. As far as we know, however, it has never been applied to a potentially power-law inelastic liquid, which is the main aim of the present work.

In Sec. II, the theoretical background of self-thinning threads of power-law inelastic fluids is described. In Sec. III the materials used in the experiments are presented. In Sec. IV the experimental setup is demonstrated. Section V is devoted to the experimental results and discussion. The conclusions are drawn in Sec. VI.

II. THEORETICAL BACKGROUND

Preliminary study of gelled propellants, oxidizers and their simulants in simple shear flow (Rahimi et al., 2001; see also the present work) showed that they can be characterized by the power-law constitutive equation. The three-dimensional power-law model is given by (Astarita and Marrucci, 1974)

\[ \tau = 2K[2 \text{tr}(\mathbf{D}^2)]^{(n-1)/2} \mathbf{D}, \]  

(1)

where \( \tau \) is the deviatoric stress tensor, \( \mathbf{D} \) is the rate-of-strain tensor, \( K \) and \( n \) are the rheological parameters of the liquid. This constitutive model is actually a formal one, and it is a good question whether it is capable of describing the behavior of real fluids, say, in both simple shear and uniaxial elongational flows, which can be defined as the bounds of the whole possible range of different flows. Some examples of such capability are known in literature (see Secs. I and V). In the present work the model is tested on gelled propellant simulants.

For a simple shear flow with shear rate \( \dot{\gamma} \) this equation reduces to the well-known Ostwald–de Waele one-dimensional power law

\[ \tau_{sh} = K \dot{\gamma}^n, \]  

(2)

(where \( \tau_{sh} \) is the shear stress), a particular case of the generalized Herschel–Bulkley model [Macosko (1994)] corresponding to the case of negligible yield stress. The shear viscosity is \( \eta_{sh} = \tau_{sh}/\dot{\gamma} \).
The rheological parameters $K$ and $n$ can be measured in simple shear. One of our aims was to test whether the rheological constitutive equation (1) is also valid for uniaxial elongational flow in threads of the gels (see Figs. 1 and 2), as well as whether the corresponding rheological parameters $K$ and $n$ would be close to those measured in simple shear. If so, the elongational behavior of the gels can be inferred from their shear behavior. The threads in the experiments were nonuniform with their cross-sectional radius slightly dependent on the axial coordinate $x$ in the middle part (cf. Fig. 2); this is reminiscent of the threads of Newtonian fluids [McKinley and Tripathy (2000)]. In the latter work it was demonstrated how the results corresponding to self-thinning of a uniform thread could be modified for the case of a slightly nonuniform one. In the present work a similar modification will be applied in the case of power-law fluids.

A thread thins and pinches under the action of the capillary forces resulting in the uniaxial elongational flow. The calculation of the thread shape is given in the Appendix, yielding the following expression for the cross-sectional diameter in the middle of a pinching thread

$$d = d_0 \left( \frac{t_s - t}{t_s} \right)^n \left( \frac{t_s^2}{d_0 \kappa} \right) \frac{1}{3^{(n-1)/2}} \frac{1}{6n(1 + \beta_1)},$$

(3)

where $\beta_1 = 0.175$, $t_s$ is the moment of pinching, and

$$\kappa = K/\sigma.$$  

(4)

The thread cross-sectional diameter at the beginning of pinching is denoted $d_0$ (shifted to $t = 0$).

For a Newtonian thread Eq. (3) reduces to

$$d = (t_s - t) \frac{\sigma}{3\mu} \frac{1}{2(1 + \beta_1)},$$

(5)

which corresponds to the corrected expression for Newtonian fluids used in McKinley and Tripathy (2000). Before the correction was introduced, Bazilevsky et al. (1990a)

![Figure 1. Cylindrical liquid thread.](image)
used Eq. (5) for Newtonian fluids with $\beta_1 = 0$, which leads to inaccurate values of $\mu$ deduced from the elongational tests.

According to Eq. (3),

$$\log \frac{d}{d_0} = n \log \left( \frac{t_s - t}{t_s} \right) + C(n, \kappa),$$

where

$$C(n, \kappa) = \log \left[ \frac{P_s}{d_0 \kappa} \frac{1}{3(\kappa - 1)\kappa} \frac{1}{6n(1 + \beta_1)} \right].$$

The experimental strategy in the present work is based on Eqs. (6) and (7). In Eq. (6), the slope of the experimental data for $d(t)$ in log–log coordinates yields the value of $n$. The value of $\kappa$ is then found using the fitted data and $C(n, \kappa)$ from Eq. (7), and the value of $K$ can be found from Eq. (4) if the surface tension $\sigma$ is measured independently. It is emphasized that due to the approximate nature of Eqs. (3), (6), and (7) their range of validity is $0.4 \leq n \leq 1$. 

**FIG. 2.** Elongational flow in a tested gel (Carbopol 0.5%) as a function of time. The snapshots were taken at 4000 fps with shutter speed 1/4000. The cross-sectional diameter of the thread (in the middle) in the first picture was $d = 1.36$ mm.
III. MATERIALS USED

Various gels were prepared to serve as propellant simulants. These water-based inert substances are used wherever the actual propellant can be dispensed with, such as for research and development of propulsion systems.

The gellants used in the work of Rahimi et al. (2001) were organic and inorganic, including polyacrylates and polysaccharides in concentrations varying between 0.3% and 5%, and silica and metal oxides between 2% and 7%. For the present study water-based gels were used with Carbopol as the only gellant adding a significant plastic component to a viscous solvent, at weight concentrations of 0.3%, 0.5%, 0.75%, and 1%. Commercial Carbopol (ETD™ 2050 Polymer) was purchased from B.F. Goodrich Co. Standard mixing procedures were applied in preparing the gels.

In the present work no solid particles were added to the gels, and their effect on the rheology is not studied.

IV. EXPERIMENTAL SETUP

The shear flow experiments were conducted following Rahimi et al. (2001). The rheological characterization was carried out using mainly shear-stress-controlled TA Instruments rotational rheometer (CSTR2000) with cone-and-plate (40 mm diameter×2°) and parallel-plate geometry. The no-slip condition during the experiments was verified: the flow field was colored with a dye, which brought out a perfect Couette velocity profile. Special procedures provided by the rheometer manufacturer for eliminating air-friction parasitic moments and inertia effects due to the rotating rheometer parts and gap variations due to thermal changes were implemented. The inaccuracy in shear stress measurements was less than 3%. A detailed description of the shear flow experiments is given in Rahimi (1999) where a similar procedure was employed.

The constitutive equations verified by means of the rotational rheometer were one-dimensional equations based on the shear projection only. The simple power-law of Eq. (2) was found to be representative.

The experimental setup used for the elongational test consists of an optical measurement unit, a fixed lower plate, and a movable upper plate which can be quickly pulled up by an electromagnet (see Fig. 3). The measurement unit contains a light source and a high-speed camera, which transmits the filament shape in digital form to a PC.

The principle of work of the present setup is illustrated in Figs. 1 and 3. A droplet of a tested fluid is placed between the plates and the movable plate is raised, stretching the droplet. As a result, a liquid thread is formed. After the movable plate has come to rest, the thread continues to thin. The flow is driven by capillary forces, the capillary pressure in the thread (the only pressure component in a thin thread) being of the order of 2σ/d. On the other hand, the capillary pressure in the end regions of the thread (see Fig. 1) is
of the order of $\sigma/R$, where $R$ is the characteristic size of these regions. Since $R \gg d/2$, there are pressure drops from the thread center toward the end regions. Therefore, the surface tension forces actually squeeze liquid out of the thread, and elongational flow (depicted by the arrows in Fig. 1) arises. The initial droplet length in the $x$ direction was about 1.1 mm, the thread length during self-thinning was kept constant at 3.5 mm; the unstretched end sections of the filaments totaled 1.75 mm. The camera was operated at a frame rate of 4000 fps and at shutter speed of 1/10 000. A sequence of snapshots at intervals of 5 ms is presented in Fig. 2. The flow arising in the present experiment was typical elongational flow.

In the present experiments three different end-plate materials—copper, brass, and glass—were used to reveal the possible effect of the material on the measurement results for all the gel simulants studied. The diameter of the end plates was 7 mm for copper, 7 mm for brass, and 18 mm for glass. The surface roughness of the copper and brass plates was $R_a = 1.8 \mu m$, whereas that of the glass plate was $R_a = 0.52 \mu m$. The need to study the differences between the three substrates is related to the fact that an insufficient wettability by a test fluid could yield almost hemispherical end regions in the thread. This phenomenon, if it happens, should be distinguished for a reliable interpretation of the experimental data [Stelter et al. (2000)].

V. EXPERIMENTAL RESULTS AND DISCUSSION

The shear stress and viscosity measured for the gels are presented in Fig. 4. They were fitted by the power law of Eq. (2), since the fluids demonstrated a clear shear-thinning behavior. The results of the shear measurements showed that the power law approximation fits the data at shear rates $\dot{\gamma}$ from 1 to 100 l/s.

The measurements of the present work yielded the value of $\sigma = 80 \times 10^{-3} \text{ kg/s}^2$ (the maximum bubble pressure method was used). For a comparable system Green et al. (1991) published the value of $\sigma = 76.3 \times 10^{-3} \text{ kg/s}^2$. The values of $n$ and $k$ obtained in the shear tests are presented in Tables I and II and compared with the results of the elongational tests discussed in the following.

For each fluid/plate-material pair at least several tens of elongational tests were conducted to obtain good statistics of the fitting parameters. In each case the filament diameter $d(t)$ was measured. Small-scale noise in the experimental data was smoothed by means of cubic splines. An example of such data is shown in Fig. 5(a). The section shown corresponds to the final stage of the thread pinching, when the effect of the initial stretching has completely vanished. Such data were replotted in log–log coordinates and fitted by Eqs. (6) and (7), yielding the corresponding values of $n$ and $k$ or $(K)$ [cf. Fig. 5(b)]. The fitting curve with the found values of $n$ and $k$ is also plotted in Fig. 5(a) as $d$ versus $t$ for comparison. It was found that the lowest values of $n$ correspond to the steepest slopes at the final section of curves $d(t)$. This was taken to mean that the tests yielding high values of $n$ involve less accurate measurements missing a number of points at the last short stage of pinching. Therefore these data were discarded and in the following we only present those data corresponding to the lowest values of $n$ for each fluid/plate-material pair. The experimental results obtained in the elongational tests for all the pairs are shown in Figs. 5(a)–5(b), 6(a)–6(c), 7(a)–7(d), and 8(a)–8(d), in which the fitting lines according to Eq. (6) are shown as well. The corresponding values of $n$ and $k$ are incorporated in Tables I and II, respectively.

The fact that the elongational data could be fitted can be interpreted as follows: the rheological behavior of the tested fluid obeys rather closely the power-law constitutive equation. It is emphasized that the dependencies $d(t)$ obtained in the present work are
FIG. 4. Shear stress and shear viscosity as functions of shear rate $\dot{\gamma}$. Experimental data are shown by symbols: crosses for shear stress and circles for shear viscosity. Straight lines 1 and 2 show shear stress and shear viscosity according to the power law fitting of the experimental data. (a) Carbopol 0.3%, (b) 0.5%, (c) 0.75%, and (d) 1%. The measurements were conducted at 25 °C. The length of time each shear stress was applied before a measurement was taken, was much more than $10^{-2}$ s which is the characteristic time of the transient flow in the present case. Typically it took 1–10 s before a point was obtained. The corresponding error bars are less than the symbols in the plot.
quite distinct from those, for example, of viscoelastic liquids. The latter are exponential according to Bazilevsky et al. (1990a, b) Stelter et al. (2000), and McKinley (2000). On the other hand, Newtonian fluids demonstrate a constant slope of the dependencies \( d(t) \) [cf. Eq. (5)] and cannot be fitted to the data in Figs. 5–8, especially in their steepest final sections.

The rate of elongation \( \dot{\varepsilon} = \frac{\partial V}{\partial x} \) is estimated for a uniform thread as \( \dot{\varepsilon} = -(2/d) d/dt. \) It is calculated using numerical differentiation (based on central differences) of the experimental data for \( d(t). \) It was found that at \( t \rightarrow t_s \) the values of \( \dot{\varepsilon} \) of the order of \( 10^2 \text{--} 10^3 \text{ l/s} \) are reached.

The measured values of \( n \) and \( \kappa \) show that for the gelled simulants studied in the present work effect of the plate material is not very large, except in the case of the 0.5% Carbopol gel where we suspect an inhomogeneity was present. Moreover, detailed examination of video records of the self-thinning process does not reveal any significant differences between the fluid behaviors on different plates.

As for the two sets of \( n \) and \( \kappa \) found in simple shear and uniaxial elongation it is seen that the values of \( n \) are fairly the same for all the fluids tested, namely about 0.4 (Table I). The values of \( \kappa \) measured in shear are lower than those measured in elongation. This probably can be attributed to lower accuracy of the elongational tests. While the disagreement between the values of \( n \) and \( \kappa \) measured in shear and elongation is not dramatic, neither is the agreement impressive. Still, there is no doubt that the elongational test brings out roughly the same pattern of rheological behavior as the simple shear test.

Jones et al. (1987) showed that deviations of rheological behavior of different fluids from the three-dimensional power law constitutive equation (1) can be characterized by an appropriately chosen Trouton ratio \( T_R \). It is based on the elongational and shear viscosities

\[
\eta_{el}(\dot{\varepsilon}) = \frac{\tau_{xx} - \tau_{yy}}{\dot{\varepsilon}} = K^3(n+1)/2 \varepsilon^{n-1},
\]

**TABLE I.** The values of \( n \).

<table>
<thead>
<tr>
<th>Test fluid content of Carbopol</th>
<th>Plate material</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Copper (elongation)</td>
</tr>
<tr>
<td>------------------------------</td>
<td>---------------------</td>
</tr>
<tr>
<td>0.3%</td>
<td>0.408</td>
</tr>
<tr>
<td>0.5%</td>
<td>0.535</td>
</tr>
<tr>
<td>0.75%</td>
<td>0.335</td>
</tr>
<tr>
<td>1%</td>
<td>0.478</td>
</tr>
</tbody>
</table>

**TABLE II.** The values of \( \kappa; [\kappa] = 10^2 \text{ s}^2/\text{m}. \)

<table>
<thead>
<tr>
<th>Test fluid content of Carbopol</th>
<th>Plate material</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Copper (elongation)</td>
</tr>
<tr>
<td>------------------------------</td>
<td>---------------------</td>
</tr>
<tr>
<td>0.3%</td>
<td>1.295</td>
</tr>
<tr>
<td>0.5%</td>
<td>0.576</td>
</tr>
<tr>
<td>0.75%</td>
<td>4.026</td>
</tr>
<tr>
<td>1%</td>
<td>3.387</td>
</tr>
</tbody>
</table>
where $\tau_{xx}$ and $\tau_{yy}$ are the axial and radial deviatoric stresses in the liquid thread. The Trouton ratio is given by

$$T_R = \frac{\eta_{d}(\dot{\gamma})}{\eta_{sh}(\sqrt{3}\dot{\gamma})},$$

and if a fluid follows the power law, Eqs. (8)–(10) yield $T_R = 3$ [Jones et al. (1987)].

The values of $T_R$ found in the present experiments are listed in Table III. They were found using the average value of $n$ for the corresponding shear and elongational tests. In spite of a scatter in the range 0.57–5.65, they are reasonably close to the value of 3 corresponding to the power law (1). On the other hand, fluids which do not follow the power law demonstrate the values of $T_R$ of the order of $10^2$–$10^3$ [Jones et al. (1987)].

Since the tested gels follow the power law closely enough, the elongational behavior can be inferred from the shear behavior. Moreover, the validity of the power law allows
us to assume that in between, when both shear and elongational components are present in a single flow field, the above law is capable of describing the rheological behavior of the gels correctly. This “universal” validity of the three-dimensional power law is quite peculiar. Still, some examples of truly power law behavior in both elongation and shear
FIG. 7. Elongation test. Filament diameter vs time. Brass plates. (a) Carbopol 0.3%; (b) 0.5%; (c) 0.75%; (d) 1%. Curves 1—experimental data, 2—fitting by Eq. (3). Error bars are also shown.
FIG. 8. Elongation test. Filament diameter vs time. Glass plates. (a) Carbopol 0.3%; (b) 0.5%; (c) 0.75%; (d) 1%. Curves 1—experimental data, 2—fitting by Eq. (3). Error bars are also shown.
were observed in the past for concentrated suspensions [Entov et al. (1980); Yarin (1993)]. The present propellant simulants yield an additional example of this kind.

VI. CONCLUSIONS

The gelled simulants studied in the present work fairly closely followed power-law constitutive equations in both simple shear and uniaxial elongational tests over two decades of shear and elongation rates. This result is of interest in the context of atomization of such gels, since elongational components are typical in flows leading to droplet detachment. The fact that the power-law behavior recorded in uniaxial elongation is compatible with the shear-thinning behavior recorded in simple shear flows of the gels, allows reliable rheological characterization of the gelled simulants by the simple shear test alone.

ACKNOWLEDGMENTS

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APPENDIX

The quasi-one-dimensional equations of continuity and momentum of power law fluids, which can be used to describe thread pinching, read [Yarin (1993)]

\[
\frac{\partial a^2}{\partial t} + \frac{\partial V a^2}{\partial x} = 0, \tag{A1}
\]

\[
\frac{\partial}{\partial x} \left[ 3a^2 a^{n} + a \right] = 0, \tag{A2}
\]

where \( t \) is time, \( x \) the axial coordinate, \( a \) the cross-sectional radius, and \( V \) the longitudinal velocity. The equations here and hereinafter are rendered dimensionless by the scales \( a_0 \) for \( a \) and \( x \), \( \alpha/\mu_{\text{eff}} \) for \( V \) and \( \mu_{\text{eff}} a_0/\alpha \) for \( t \), where the effective viscosity \( \mu_{\text{eff}} = 3^{(n-1)/2n} K^{1/n} (\alpha/a_0)^{1-1/n} \); \( a_0 \) is the radius at \( t = 0 \).

Close to the thread pinching the solution is expected to be self-similar, in analogy to the Newtonian case studied in Papageorgiou (1995). The solution is then sought in the following form:

\[
a = (t_s-t)^\mu f(\xi), \tag{A3}
\]

\[
V = (t_s-t)^{\beta-1} g(\xi), \tag{A4}
\]

\begin{table}[h]
\centering
\caption{The values of Trouton ratio \( T_R \).}
\begin{tabular}{|c|c|c|c|}
\hline
Test fluid content of Carbopol & Copper & Brass & Glass \\
\hline
0.3\% & 2.49 & 3.22 & 2.14 \\
0.5\% & 1.03 & 5.65 & 0.57 \\
0.75\% & 2.73 & 2.53 & 1.8 \\
1\% & 5.24 & 1.53 & 2.05 \\
\hline
\end{tabular}
\end{table}
where $\xi = x/(t_s - t)^{\beta}$, and $t_s$ is the moment when the thread vanishes. The exponent $\beta$ should be found as an eigenvalue of the problem.

Substituting Eqs. (A3) and (A4) in (A1) and (A2), we obtain the following set of equations for the functions $f$ and $g$:

\[ (g + \beta \xi) \frac{df}{d\xi} + \left[ \frac{1}{2} \frac{dg}{d\xi} - n \right] f = 0, \tag{A5} \]

\[ \frac{d}{d\xi} \left[ 3f^2 \left( \frac{dg}{d\xi} \right)^n + f \right] = 0. \tag{A6} \]

The local behavior of the solutions near the pinching point $\xi_0$ should be such that a singularity in Eq. (A5) is obviated, namely

\[ f(\xi) = f_0 + (\xi - \xi_0)f_2 + \cdots, \tag{A7} \]

\[ g(\xi) = -\beta \xi_0 + 2n(\xi - \xi_0) + (\xi - \xi_0)^3 g_3 + \cdots, \tag{A8} \]

where $f_0$, $f_2$, $g_3$, etc., are the coefficients.

In the leading order term $dg/d\xi = 2n$, whereas $(dg/d\xi)^{n-1} = (2n)^{n-1} = F(n)$. This function is close to unity in the range $0.4 \leq n \leq 1$, since $F(1) = F(1/2) = 1$ [the maximum deviation of $F(n)$ from unity is about 10% for $0.4 \leq n \leq 1$]. Therefore, Eq. (A6) could be approximately replaced by

\[ \frac{d}{d\xi} \left[ 3f^2 \frac{dg}{d\xi} + f \right] = 0 \tag{A9} \]

with all the results valid for $0.4 \leq n \leq 1$.

Substituting Eqs. (A7) and (A8) in (A5) and (A9), we find

\[ f_0 = \frac{1}{12(n + \beta)}. \tag{A10} \]

Equation (A9) can be integrated once, which yields

\[ \frac{dg}{d\xi} = -\frac{1}{3f} + \frac{k}{f^2} \tag{A11} \]

with

\[ k = \frac{3n + 2\beta}{72(n + \beta)^2} \tag{A12} \]

given by the local solution (A10).

Integrating Eq. (A11) once again, in analogy to Papageorgiou, we find the global condition which allows calculation of the eigenvalue $\beta$,

\[ k = \frac{1}{3} \int_0^\infty \frac{d(\xi - \xi_0)}{f}. \tag{A13} \]

Therefore the set of equations (A5) and (A11) should be solved for any $n$ from the range $0.4 \leq n \leq 1$ to find the value of $\beta$ such that Eq. (A13) would be satisfied.

Following Papageorgiou, we introduce the new function and variable

\[ G = g + \beta \xi_0, \tag{A14} \]
\[ \zeta = \xi - \xi_0. \]  
\[ \text{(A15)} \]

Then Eqs. (A5) and (A11) take the form

\[ \frac{df}{d\xi} = -\left(\frac{1}{2} \frac{dG/d\xi - n}{G + \beta \xi}\right) f, \]  
\[ \text{(A16)} \]

\[ \frac{dG}{d\xi} = -\frac{1}{3f + \beta \xi^2}. \]  
\[ \text{(A17)} \]

We next introduce the transformation

\[ f_1 = nf, \quad G_1 = \frac{G}{n}, \quad \beta = \beta_1 n. \]  
\[ \text{(A18)} \]

Then Eqs. (A16), (A17), (A12), and (A13) take the following form:

\[ \frac{df_1}{d\zeta_1} = -\left(\frac{1}{2} \frac{dG_1/d\zeta_1 - 1}{G_1 + \beta_1 \zeta_1}\right) f_1, \]  
\[ \text{(A19)} \]

\[ \frac{dG_1}{d\zeta_1} = \frac{1}{3f_1} + \frac{k_1}{f_1^2}, \]  
\[ \text{(A20)} \]

\[ k_1 = \frac{3 + 2\beta_1}{72(1 + \beta_1)^2}, \]  
\[ \text{(A21)} \]

\[ k_1 = \frac{1}{3} \frac{f_0^6 d\zeta/f_1^2}{f_0^6 d\zeta/f_1^2}, \]  
\[ \text{(A22)} \]

which was already solved numerically by Papageorgiou with the same boundary conditions. He found \( \beta_1 = 0.175 \). Therefore, according to Eqs. (A10) and (A18) the pinching factor \( f_0 \) is given by

\[ f_0 = \frac{1}{12n(1 + \beta_1)} = \frac{0.0709}{n}, \]  
\[ \text{(A23)} \]

and Eqs. (A3) and (A23) determine the dimensional expression (3).


Maconsko, C.W., Rheology, Principles, Measurements and Applications (VCH, New York, 1994).


